New results on equilibria of fuzzy abstract economies

Monica Patriche

University of Bucharest, Faculty of Mathematics and Computer Science, Academiei 14 Street,010014Bucharest,Romania. E-mail:monica.patriche@yahoo.com

Abstract

We obtain new equilibrium theorems for fuzzy abstract economies with correspondences being w-upper semicontinuous or having e-USS-property.

Keywords: w-upper semicontinuous correspondences, correspondences with e-USS-property, fuzzy abstract economy, fuzzy equilibrium.

JEL Classification: C72

1. Introduction

Since the theory of fuzzy sets, initiated by Zadeh [23], was considered as a framework for phenomena which can not be characterized precisely, a lot of extensions of game theory results have been established. So, many theorems concerning fuzzy equlibrium existence for fuzzy abstract economies were obtained. In [10] the authors introduced the concept of a fuzzy game and proved the existence of equlibrium for 1-person fuzzy game. Also, the existence of equilibrium points of fuzzy games was studied in [8], [9], [11],[12], [13], [14], [15], [16], [19]. Fixed point theorems for fuzzy mappings were proven in [1], [2], [7], [10].

There are several generalizations of the classical model of abstract economy proposed in his pioneering works by Debreu [4] or later by Shafer and Sonnenschein [18], Yannelis and Prahbakar [21]. In this paper we consider a fuzzy extension of Yuan's model of the abstract economy [22] and we prove the existence of fuzzy equilibrium of fuzzy abstract economies in several cases. We define two types of correspondences: w-upper semicontinuous correspondences and correspondences that have e-USS-property. By using a fixed point theorem for w-upper semicontinuous correspondences [17], we prove our first

theorem of equilibrium existence for abstract economies having w-upper semicontinuous constraint and preference correspondences. The considered fixed theorem is a Wu like result [20] and generalizes the Himmelberg's fixed point theorem in [6]. On the other hand, we use a technique of approximation to prove an equilibrium existence theorem for set valued maps having e-USS-property.

The paper is organized in the following way: Section 2 contains preliminaries and notation. The weakly upper semicontinuous correspondences with respect to a set and the fixed point theorem are presented in Section 3. The equilibrium theorems are stated in Section 4.

2. Preliminaries and notation

Throughout this paper, we shall use the following notations and definitions:

Let A be a subset of a topological space X. $F(A)$ denotes the family of all nonempty finite subsets of A. 2^A denotes the family of all subsets of A. cl$A$ denotes the closure of A in X. If A is a subset of a vector space, co$A$ denotes the convex hull of A. If F, $G : X \to 2^Y$ are correspondences, then co$G$, cl$G$, $G \cap F : X \to 2^Y$ are correspondences defined by $(\text{co}G)(x) = \text{co}G(x)$, $(\text{cl}G)(x) = \text{cl}G(x)$ and $(G \cap F)(x) = G(x) \cap F(x)$ for each $x \in X$, respectively. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$.

The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y}\text{Gr}T\}$ (the set $\text{cl}_{X \times Y}\text{Gr}(T)$ is called the adherence of the graph of T). It is easy to see that $\text{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Notation. Let E and F be two Hausdorff topological vector spaces and $X \subset E$, $Y \subset F$ be two nonempty convex subsets. We denote by $\mathcal{F}(Y)$ the collection of fuzzy sets on Y. A mapping from X into $\mathcal{F}(Y)$ is called a fuzzy mapping. If $F : X \to \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denoted by F_x in this sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x.

A fuzzy mapping $F : X \to \mathcal{F}(Y)$ is called convex, if for each $x \in X$, the fuzzy set F_x on Y is a fuzzy convex set, i.e., for any $y_1, y_2 \in Y$, $t \in [0, 1]$, $F_x(ty_1 + (1 - t)y_2) \geq \min\{F_x(y_1), F_x(y_2)\}$.

In the sequel, we denote by

$(A)_q = \{y \in Y : A(y) \geq q\}$, $q \in [0, 1]$ the q-cut set of $A \in \mathcal{F}(Y)$.

Definition 1. *Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be upper semicontinuous if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. T is said to be almost upper semicontinuous if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset$clV for each $y \in U$.*

Lemma 1. *(Lemma 3.2, pag. 94 in [24])Let X be a topological space, Y be a topological linear space, and let $S : X \to 2^Y$ be an upper semicontinuous correspondence with compact values. Assume that the sets $C \subset Y$ and $K \subset Y$ are closed and respectively compact. Then $T : X \to 2^Y$ defined by $T(x) = (S(x) + C) \cap K$ for all $x \in X$ is upper semicontinuous.*

Lemma 2 is a version of Lemma 1.1 in [22] (for $D = Y$, we obtain Lemma 1.1 in [22]). For the reader's convenience, we include its proof below.

Lemma 2. *Let X be a topological space, Y be a nonempty subset of a locally convex topological vector space E and $T : X \to 2^Y$ be a correspondence. Let ß be a basis of neighbourhoods of 0 in E consisting of open absolutely convex symmetric sets. Let D be a compact subset of Y. If for each $V \in$ß, the correspondence $T^V : X \to 2^Y$ is defined by $T^V(x) = (T(x) + V) \cap D$ for each $x \in X$, then $\cap_{V \in ß}\overline{T^V}(x) \subseteq \overline{T}(x)$ for every $x \in X$.*

Proof. Let x and y be such that $y \in \cap_{V \in ß}\overline{T^V}(x)$ and suppose, by way of contradiction, that $y \notin \overline{T}(x)$. This means that $(x, y) \notin$cl Gr T, so that there exists an open neighborhood U of x and $V \in$ß such that:

$(U \times (y + V))\cap$Gr $T = \emptyset$. (1)

Choose $W \in$ß such that $W - W \subseteq V$ (e.g. $W = \frac{1}{2}V$). Since $y \in T^W(x)$, then $(x, y) \in$cl Gr T^W, so that

$$(U \times (y + W)) \cap \text{Gr } T^W \neq \emptyset.$$

There are some $x' \in U$ and $w' \in W$ such that $(x', y + w') \in$Gr T^W, i.e. $y + w' \in T^W(x')$. Then, $y + w' \in D$ and $y + w' = y' + w''$ for some $y' \in T(x')$ and $w'' \in W$. Hence, $y' = y + (w' - w'') \in y + (W - W) \subseteq y + V$, so that $T(x') \cap (y + V) \neq \emptyset$. Since $x' \in U$, this means that $(U \times (y + V))\cap$Gr $T \neq \emptyset$, contradicting (1). □

3. Weakly upper semicontinuous correspondences with respect to a set

We introduce the following definitions.

Let X be a topological space, Y be a nonempty subset of a topological vector space E and D be a subset of Y.

Definition 2. *The correspondence $T : X \to 2^Y$ is said to be w-upper semicontinuous (weakly upper semicontinuous) with respect to the set D if there exists a basis ß of open symmetric neighborhoods of 0 in E such that, for each $V \in$ß, the correspondence T^V is upper semicontinuous.*

Definition 3. *The correspondence $T : X \to 2^Y$ is said to be almost w-upper semicontinuous (almost weakly upper semicontinuous) with respect to the set D if there exists a basis ß of open symmetric neighborhoods of 0 in E such that, for each $V \in$ß, the correspondence $\overline{T^V}$ is upper semicontinuous.*

Example 1. *Let $T_1 : (0, 2) \to 2^{(0,2)}$ be defined by $T_1(x) = \begin{cases} (0,1) & \text{if } x \in (0,1]; \\ [1,2) & \text{if } x \in (1,2). \end{cases}$*

T_1 and $T_1 \cap \{1\} = \begin{cases} \phi & \text{if } x \in (0,1]; \\ \{1\} & \text{if x} \in (1,2) \end{cases}$ are not upper semicontinuous on $(0, 2)$, but T_1 is w-upper semicontinuous with respect to D and it is also almost w-upper semicontinuous with respect to D.

We also define the dual w-upper semicontinuity with respect to a compact set.

Definition 4. *Let $T_1, T_2 : X \to 2^Y$ be correspondences. The pair (T_1, T_2) is said to be dual almost w-upper semicontinuous (dual almost weakly upper semicontinuous) with respect to the set D if there exists a basis ß of open symmetric neighborhoods of 0 in E such that, for each $V \in$ß, the correspondence $\overline{T^V_{(1,2)}} : X \to 2^D$ is lower semicontinuous, where $T^V_{(1,2)} : X \to 2^D$ is defined by $T^V_{(1,2)}(x) = (T_1(x) + V) \cap T_2(x) \cap D$ for each $x \in X$.*

Example 2. *Let $D = [1, 2]$, $T_1 : (0, 2) \to 2^{[1,4]}$ be the correspondence defined by*

$$T_1(x) = \begin{cases} [2-x,2], & \text{if } x \in (0,1); \\ \{4\} & \text{if } \quad x = 1; \\ [1,2] & \text{if } \quad x \in (1,2). \end{cases}$$

and $T_2 : (0,2) \to 2^{[2,3]}$ be the correspondence defined by

$$T_2(x) = \begin{cases} [2,3], & \text{if } x \in (0,1]; \\ \{2\} & \text{if } \quad x \in (1,2); \end{cases} .$$

The correspondence T_1 is not upper semicontinuous on $(0,2)$, but $\overline{T^V_{(1,2)}}$ is upper semicontinuous and has nonempty values.

We conclude that the pair (T_1, T_2) is dual almost w-upper semicontinuous with respect to D.

We obtain the following fixed point theorem which generalizes Himmelberg's fixed point theorem in [6]:

Theorem 3. *(see [17])Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i be a nonempty compact convex subset of X_i and $S_i, T_i : X := \prod_{i\in I} X_i \to 2^{X_i}$ be two correspondences with the following conditions:*

1) *for each $x \in X$, $\overline{S}_i(x) \subseteq T_i(x)$.*

2) *S_i is almost w-upper semicontinuous with respect to D_i and $\overline{S_i^{V_i}}$ is convex nonempty valued for each absolutely convex symmetric neighborhood V_i of 0 in E_i.*

Then there exists $x^* \in D := \prod_{i\in I} D_i$ such that $x_i^* \in T_i(x^*)$ for each $i \in I$.

4. Existence of fuzzy equilibrium for fuzzy abstract economies

4.1. The model of a fuzzy abstract economy

In this section we describe the fuzzy equilibrium for a fuzzy extension of Yuan's model of abstract economy [22]. We prove the existence of fuzzy equilibrium of abstract fuzzy economies in several cases.

Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a non-empty topological vector space representing the set of actions and define $X := \prod_{i\in I} X_i$; let A_i, $B_i : X \to \mathcal{F}(X_i)$ be the constraint fuzzy correspondences and $P_i : X \to \mathcal{F}(X_i)$ the preference fuzzy correspondence, $a_i, b_i : X \to (0,1]$ fuzzy constraint functions and $p_i : X \to (0,1]$ fuzzy preference function.

Let denote $A'_i,\ B'_i, P'_i : X \to 2^{X_i}$, defined by $A'_i(x) = (A_{i_x})_{a_i(x)}$, $B'_i(x) = (B_{i_x})_{b_i(x)}$ and $P'_i(x) = (P_{i_x})_{b_i(x)}$.

Definition 4. A fuzzy *abstract economy* is defined as an ordered family $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$.

If $A_i, B_i, P_i : X \to 2^{Y_i}$ are classical correspondences, then the previous definition can be reduced to the standard definition of abstract economy due to Yuan [22].

Definition 5. A *fuzzy equilibrium* for Γ is defined as a of point $x^* \in X$ such that for each $i \in I$, $x^*_i \in \overline{B'}(x^*)$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$, where $(A_{i_{x^*}})_{a_i(x^*)} = \{z \in Y_i : A_{i_{x^*}}(z) \geq a_i(x^*)\}$, $(B_{i_{x^*}})_{b_i(x^*)} = \{z \in Y_i : B_{i_{x^*}}(z) \geq b_i(x^*)\}$, $(P_{i_{x^*}})_{p_i(x^*)} = \{z \in Y_i : P_{i_{x^*}}(z) \geq p_i(x^*)\}$.

4.2. Equilibria existence

As an application of the fixed point Theorem 1, we have the following result.

Theorem 4. *Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ be a fuzzy abstract economy such that for each $i \in I$ the following conditions are fulfilled:*

1) X_i be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E_i and D_i is a nonempty compact convex subset of X_i;

2) A_i, P_i and B_i are such that each $(B_{i_x})_{b_i(x)}$ is a nonempty convex subset of X_i, $(A_{i_x})_{a_i(x)}, (P_{i_x})_{p_i(x)}$ are convex and $(A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} \subset (B_{i_x})_{b_i(x)}$ for each $x \in X$;

3) the set $W_i = \left\{x \in X : (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} \neq \emptyset\right\}$ is open in X.

4) the correspondence $H_i : X \to 2^{X_i}$ defined by $H_i(x) = (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)}$ for each $x \in X$ is almost w-upper semicontinuous with respect to D_i on W_i and $\overline{H_i^{V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighborhood V_i of 0 in E_i;

5) the correspondence $x \to \underline{(B_{i_x})_{b_i(x)}} : X \to 2^{X_i}$ is almost w-upper semicontinuous with respect to D_i and $\overline{B_i^{V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighborhood V_i of 0 in E_i, where $B_i^{V_i} : X \to 2^{X_i}$ is defined by $B_i^{V_i}(x) = ((B_{i_x})_{b_i(x)} + V_i) \cap D_i$;

6) for each $x \in X$, $x_i \notin \overline{H}_i(x)$;

Then there exists a fuzzy equilibrium point $x^* \in D = \prod_{i \in I} D_i$ *such that for each* $i \in I$, $x_i^* \in \overline{B}_i'(x^*)$ *and* $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$.

Proof. Let $i \in I$. By condition (3) we know that W_i is open in X.

Let's define $T_i : X \to 2^{X_i}$ by $T_i(x) = \begin{cases} (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)}, & \text{if } x \in W_i, \\ (B_{i_x})_{b_i(x)}, & \text{if } x \notin W_i \end{cases}$ for each $x \in X$.

Then $T_i : X \to 2^{X_i}$ is a correspondence with nonempty convex values. We shall prove that $T_i : X \to 2^{D_i}$ is almost w-upper semicontinuous with respect to D_i. Let ß$_i$ be a basis of open absolutely convex symmetric neighborhoods of 0 in E_i and let ß$=\prod_{i \in I}$ß$_i$.

For each $V = (V_i)_{i \in I} \in \prod_{i \in I}$ß$_i$, for each $x \in X$, let for each $i \in I$

$F_i^{V_i}(x) = ((A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} + V_i) \cap D_i$ and

$T_i^{V_i}(x) = \begin{cases} F_i^{V_i}(x), & \text{if } x \in W_i, \\ B_i^{V_i}(x), & \text{if } x \notin W_i. \end{cases}$

For each open set V_i' in D_i, the set

$$\left\{x \in X : \overline{T_i^{V_i}}(x) \subset V_i'\right\} =$$
$$= \left\{x \in W_i : \overline{F_i^{V_i}}(x) \subset V_i'\right\} \cup \left\{x \in X \setminus W_i : \overline{B_i^{V_i}}(x) \subset V_i'\right\}$$
$$= \left\{x \in W_i : \overline{F_i^{V_i}}(x) \subset V_i'\right\} \cup \left\{x \in X : \overline{B_i^{V_i}}(x) \subset V_i'\right\}.$$

According to condition (4), the set $\left\{x \in W_i : \overline{F_i^{V_i}}(x) \subset V_i'\right\}$ is open in X. The set $\left\{x \in X : \overline{B_i^{V_i}}(x) \subset V_i'\right\}$ is open in X because $\overline{B_i^{V_i}}$ is upper semicontinuous.

Therefore, the set $\left\{x \in X : \overline{T_i^{V_i}}(x) \subset V_i'\right\}$ is open in X. It shows that $\overline{T_i^{V_i}} : X \to 2^{D_i}$ is upper semicontinuous. According to Theorem 1, there exists $x^* \in D = \prod_{i \in I} D_i$ such that $x^* \in \overline{T}_i(x^*)$, for each $i \in I$. By condition (5) we have that $x_i^* \in \overline{B_i'}(x^*)$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$ for each $i \in I$.□

Theorem 5 deals with abstract economies which have dual w-upper semicontinuous pairs of correspondences.

Theorem 5. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ *be a fuzzy abstract economy such that for each* $i \in I$ *the following conditions are fulfilled:*

1) X_i be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E_i and D_i is a nonempty compact convex subset of X_i;

2) A_i, P_i, B_i are such that each $(B_{i_x})_{b_i(x)}$ is a convex subset of X_i, $(P_{i_x})_{p_i(x)} \subset D_i$ and $(A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} \subset (B_{i_x})_{b_i(x)}$ for each $x \in X$;

3) the set $W_i = \left\{x \in X : (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} \neq \emptyset\right\}$ is open in X.

4) the pair $x \to ((A_{i_x})_{a_i(x)|\mathrm{cl}W_i}, (P_{i_x})_{p_i(x)|\mathrm{cl}W_i})$ is dual almost w-upper semicontinuous with respect to D_i, the correspondence $x \to (B_{i_x})_{b_i(x)} : X \to 2^{X_i}$ is almost w-upper semicontinuous with respect to D_i;

5) if $T_{i,V_i} : X \to 2^{X_i}$ is defined by $T_{i,V_i}(x) = ((A_{i_x})_{a_i(x)}+V_i) \cap D_i \cap (P_{i_x})_{p_i(x)}$ for each $x \in X$ and $B_i^{V_i} : X \to 2^{X_i}$ is defined by $B_i^{V_i}(x) = ((B_{i_x})_{b_i(x)}+V_i) \cap D_i$ for each $x \in X$, then the correspondences $\overline{B_i^{V_i}}$ and $\overline{T_{i,V_i}}$ are nonempty convex valued for each open absolutely convex symmetric neighborhood V_i of 0 in E_i;

6) for each $x \in X$, $x_i \notin \overline{P_i'}(x)$.

Then there exists a fuzzy equilibrium point $x^* \in D = \prod_{i \in I} D_i$ such that for each $i \in I$, $x_i^* \in \overline{B_i'}(x^*)$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$.

Proof. For each $i \in I$, let $ß_i$ denote the family of all open absolutely convex symmetric neighborhoods of zero in E_i and let $ß = \prod_{i \in I} ß_i$. For each $V = \prod_{i \in I} V_i \in \prod_{i \in I} ß_i$, for each $i \in I$, let

$A_i^{V_i}, S_i^{V_i} : X \to 2^{X_i}$ be defined by

$A_i^{V_i}(x) = ((A_{i_x})_{b_i(x)} + V_i) \cap D_i$ for each $x \in X$ and

$$S_i^{V_i}(x) = \begin{cases} T_{i,V_i}(x), & \text{if } x \in W_i, \\ B_i^{V_i}(x), & \text{if } x \notin W_i, \end{cases}$$

$\overline{S_i^{V_i}}$ has closed values. Next, we shall prove that $\overline{S_i^{V_i}} : X \to 2^{D_i}$ is upper semicontinuous.

For each open set V' in D_i, the set

$$\left\{x \in X : \overline{S_i^{V_i}}(x) \subset V'\right\} =$$
$$= \left\{x \in W_i : \overline{T_{i,V_i}}(x) \subset V'\right\} \cup \left\{x \in X \setminus W_i : \overline{B_i^{V_i}}(x) \subset V'\right\}$$
$$= \left\{x \in W_i : \overline{T_{i,V_i}}(x) \subset V'\right\} \cup \left\{x \in X : \overline{B_i^{V_i}}(x) \subset V'\right\}.$$

We know that the correspondence $\overline{T_{i,V_i}}(x)_{|W_i} : W_i \to 2^{D_i}$ is upper semicontinuous. The set $\left\{x \in W_i : \overline{T_{i,V_i}}(x) \subset V'\right\}$ is open in X. Since $\overline{B_i^{V_i}}(x)$:

$X \to 2^{D_i}$ is upper semicontinuous, the set $\{x \in X : \overline{B_i^{V_i}}(x)\} \subset V'$ is open in X and therefore, the set $\left\{x \in X : \overline{S_i^{V_i}}(x) \subset V'\right\}$ is open in X. It proves that $\overline{S_i^{V_i}} : X \to 2^{D_i}$ is upper semicontinuous. According to Himmelberg's Theorem, applied for the correspondences $\overline{S_i^{V_i}}$, there exists a point $x_V^* \in D = \prod_{i\in I} D_i$ such that $(x_V^*)_i \in S_i^{V_i}(x_V^*)$ for each $i \in I$. By condition (5), we have that $(x_V^*)_i \notin \overline{P_i'}(x_V^*)$, hence, $(x_V^*)_i \notin \overline{A_i^{V_i}}(x_V^*) \cap \overline{P_i'}(x_V^*)$.
We also have that $\mathrm{clGr}(T_{i,V_i}) \subseteq \mathrm{clGr}(A_i^{V_i}) \cap \mathrm{clGr}P_i'$. Then $\overline{T_{i,V_i}}(x) \subseteq \overline{A_i^{V_i}}(x) \cap \overline{P_i'}(x)$ for each $x \in X$. It follows that $(x_V^*)_i \notin \overline{T_{i,V_i}}(x_V^*)$. Therefore, $(x_V^*)_i \in \overline{B_i^{V_i}}(x_V^*)$.

For each $V = (V_i)_{i\in I} \in \prod_{i\in I}ß_i$, let's define $Q_V = \cap_{i\in I}\{x \in D : x \in \overline{B_i^{V_i}}(x)$ and $(A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} = \emptyset\}$.

Q_V is nonempty since $x_V^* \in Q_V$, and it is a closed subset of D according to (3). Then, Q_V is nonempty and compact.

Let ß$=\prod_{i\in I}ß_i$. We prove that the family $\{Q_V : V \in ß\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of ß and let $V^{(k)} = \prod_{i\in I} V_i^{(k)}{}_{i\in I}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V \in \prod_{i\in I}ß_i$. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Since D is compact and the family $\{Q_V : V \in ß\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in ß\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in ß\}$, then for each $V \in ß$,

$x^* \in \cap_{i\in I}\left\{x^* \in D : x_i^* \in \overline{B_i^{V_i}}(x^*) \text{ and } (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} = \emptyset)\right\}.$

Hence, $x_i^* \in \overline{B_i^{V_i}}(x^*)$ for each $V \in ß$ and for each $i \in I$. According to Lemma 2, we have that $x_i^* \in \overline{(B_i')}(x^*)$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$ for each $i \in I$. $\square$

We now introduce the following concept, which also generalizes the concept of lower semicontinuous correspondences.

Definition 5. *Let X be a non-empty convex subset of a topological linear space E, Y be a non-empty set in a topological space and $K \subseteq X \times Y$.*

The correspondence $T : X \times Y \to 2^X$ has the e-USCS-property (e-upper semicontinuous selection property) on K, if for each absolutely convex neighborhood V of zero in E, there exists an upper semicontinuous correspondence with convex values $S^V : X \times Y \to 2^X$ such that $S^V(x, y) \subset T(x, y) + V$ and $x \notin \mathrm{cl}S^V(x, y)$ for every $(x, y) \in K$.

The following theorem is an equilibrium existence result for economies with constraint correspondences having e-USCS-property.

Theorem 6. *Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ be a fuzzy abstract economy such that for each $i \in I$ the following conditions are fulfilled:*

1) X_i be a non-empty compact convex subset of a locally convex Hausdorff space E_i;

(2) the correspondence $x \to cl(B_{i_x})_{b_i(x)} : X \to 2^{X_i}$ is upper semicontinuous with non-empty convex values;

(3) the set $W_i : = \left\{x \in X : (A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} \neq \emptyset\right\}$ is open;

(3) the correspondence $x \to cl((A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)}) : X \to 2^{X_i}$ has the e-USCS-property on W_i.

Then there exists an equilibrium point $x^* \in X$ for Γ, i.e., for each $i \in I$, $x_i^* \in \overline{B'}(x^*)$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$.

Proof. For each $i \in I$, let ß$_i$ denote the family of all open convex neighborhoods of zero in E_i. Let $V = (V_i)_{i\in I} \in \prod_{i\in I}$ß$_i$. Since the correspondence $x \to \mathrm{cl}((A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)})$ has the e-USCS-property on W_i, it follows that there exists an upper semicontinuous correspondence $F_i^{V_i} : X \to 2^{X_i}$ such that $F_i^{V_i}(x) \subset \mathrm{cl}((A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)}) + V_i$ and $x_i \notin \mathrm{cl}F_i^{V_i}(x)$ for each $x \in W_i$.

Define the correspondence $T_i^{V_i} : X \to 2^{X_i}$, by

$$T_i^{V_i}(x) := \begin{cases} \mathrm{cl}\{F_i^{V_i}(x)\}, & \text{if } x \in W_i, \\ \mathrm{cl}((B_{i_x})_{b_i(x)} + V_i) \cap X_i, & \text{if } x \notin W_i; \end{cases}$$

$B_i^{V_i} : X \to 2^{X_i}$, $B_i^{V_i}(x) = \mathrm{cl}((B_{i_x})_{b_i(x)} + V_i) \cap X_i = (\mathrm{cl}(B_{i_x})_{b_i(x)} + \mathrm{cl}V_i) \cap X_i$ is upper semicontinuous by Lemma 1.

Let U be an open subset of X_i, then

$$\begin{aligned} U' &:= \{x \in X \mid T_i^{V_i}(x) \subset U\} \\ &= \{x \in W_i \mid T_i^{V_i}(x) \subset U\} \cup \{x \in X \setminus W_i \mid T_i^{V_i}(x) \subset U\} \\ &= \left\{x \in W_i \mid \mathrm{cl}F_i^{V_i}(x) \subset U\right\} \cup \left\{x \in X \mid (\mathrm{cl}(B_{i_x})_{b_i(x)} + \overline{V_i}) \cap X_i \subset U\right\} \end{aligned}$$

U' is an open set, because W_i is open, $\left\{x \in W_i \mid \mathrm{cl}F_i^{V_i}(x) \subset U\right\}$ open since $\mathrm{cl}F_i^{V_i}(x)$ is an upper semicontinuous map on W_i and the set $\{x \in X \mid$

$(\mathrm{cl}(B_{i_x})_{b_i(x)} + \mathrm{cl}V_i) \cap X_i \subset U\}$ is open since $(\mathrm{cl}(B_{i_x})_{b_i(x)} + \mathrm{cl}V_i) \cap X_i$ is u.s.c. Then $T_i^{V_i}$ is upper semicontinuous on X and has closed convex values.

Define $T^V : X \to 2^X$ by $T^V(x) := \prod_{i\in I} T_i^{V_i}(x)$ for each $x \in X$.

T^V is an upper semicontinuous correspondence and has also non-empty convex closed values.

Since X is a compact convex set, by Fan's fixed-point theorem [5], there exists $x_V^* \in X$ such that $x_V^* \in T^V(x_V^*)$, i.e., for each $i \in I$, $(x_V^*)_i \in T_i^{V_i}(x_V^*)$. If $x_V^* \in W_i$, $(x_V^*)_i \in \mathrm{cl}F_i^{V_i}(x_V^*)$, which is a contradiction.

Hence, $(x_V^*)_i \in \mathrm{cl}((B_{i_{x_V^*}})_{b_i(x_V^*)} + V_i) \cap X_i$ and $(A_{i_{x_V^*}})_{a_i(x_V^*)} \cap (P_{i_{x_V^*}})_{p_i(x_V^*)} = \emptyset$, i.e. $x_V^* \in Q_V$ where

$Q_V = \cap_{i\in I}\{x \in X : x_i \in \mathrm{cl}((B_{i_x})_{b_i(x)} + V_i) \cap X_i$ and $(A_{i_x})_{a_i(x)} \cap (P_{i_x})_{p_i(x)} = \emptyset\}$.

Since W_i is open, Q_V is the intersection of non-empty closed sets, therefore it is non-empty, closed in X.

We prove that the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of $\prod_{i\in I} ß_i$ and let $V^{(k)} = (V_i^{(k)})_{i\in I}$, $k = 1, ...n$. For each $i \in I$, let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V = (V_i)_{i\in I} \in \prod_{i\in I} ß_i$. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Since X is compact and the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in \prod_{i\in I} ß_i\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in \prod_{i\in I} ß_i\}$, then for each $i \in I$ and each $V_i \in ß_i$, $x_i^* \in \mathrm{cl}((B_{i_{x^*}})_{b_i(x^*)} + V_i) \cap X_i$ and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)} = \emptyset$; but then $x_i^* \in \overline{B_i'}(x^*)$ from Lemma 2 and $(A_{i_{x^*}})_{a_i(x^*)} \cap (P_{i_{x^*}})_{p_i(x^*)}$for each $i \in I$ so that x^* is an equilibrium point of Γ in X. $\square$

References

[1] D. Butnariu, *Fixed points for fuzzy mappings,* Fuzzy Sets and Systems **7** (1982), 191-207.

[2] S.-S. Chang, *Fixed degree for fuzzy mapping and a generalization of Ky Fan's theorem,* Fuzzy Sets and Systems **24** (1987), 103-112.

[3] S.-S. Chang and K. K. Tan, *Equilibria and maximal elements of abstract fuzzy economies and qualitative fuzzy games,* Fuzzy Sets and Systems **125** (2002), 389-399.

[4] G. Debreu, *A social equilibrium existence theorem,* Proc. Nat. Acad. Sci. **38** (1952), 886-893.

[5] K. Fan,1961. *A generalization of Tychonoff's fixed point theorem,* Math. Ann. 142, 305-310.

[6] C. J. Himmelberg, 1972. *Fixed points of compact multifunctions.* J. Math. Anal. Appl. 38, 205-207.

[7] S. Heilpern, *Fuzzy mappings and fixed point theorem,* J. Math. Anal. Appl. **83** (1981) 566-569.

[8] N.-J. Huang, *A new equilibrium existence theorem for abstract fuzzy economies,* Appl. Math. Letters **12** (1999), 1-5.

[9] N.-J. Huang, *Existence of equilibrium for generalized abstract fuzzy economies,* Fuzzy Sets and Systems **117** (2001), 151-156.

[10] W. K. Kim and K. H. Lee, *Fuzzy fixed point and existence of equilibria of fuzzy games,* J. Fuzzy Math. **6** (1998), 193-202.

[11] W. K. Kim and K. H. Lee, *Existence of equilibria in generalized fuzzy games,* J. Chungcheong Math. Soc. **12** (1999), 53-61.

[12] W. K. Kim and K. H. Lee, *Generalized fuzzy games and fuzzy equilibria,* Fuzzy Sets and Systems **122** (2001), 293-301.

[13] M. Patriche, *Equilibria for free abstract fuzzy economies,* An. St. Ovidius University of Constanta, Volume 17(2), 2009, 143-154.

[14] M. Patriche, *Abstract fuzzy economies and fuzzy equilibrium pairs,* Mathematical Reports, 12(62), 4 (2010), 373 – 382

[15] M. Patriche, *Existence of fuzzy equilibria for fuzzy abstract economies with Q'-majorized correspondences,* Recent Researches in Neural Networks, Fuzzy Systems, Evolutionary Computing and Automation, International Conference on Fuzzy Systems, 11 – 13 April 2011, Brasov 2011.

[16] M. Patriche, *Equilibrium in games and competitive economies,* The Publishing House of the Romanian Academy, 2011

[17] M. Patriche and C. Hervés-Beloso, *A fixed point theorem and equilibria of abstract economies with w-upper semicontinuous correspondences,* Working paper, http://webs.uvigo.es/rgea/workingpapers/rgea213.pdf

[18] W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences,* Journal of Mathematical Economics **2** (1975), 345-348.

[19] L Wang, Y Je Cho, N Huang, *The robustness of generalized abstract fuzzy economies in generalized convex spaces,* Fuzzy Sets and Systems, 176 (1) (2011), 56-63.

[20] X. Wu, *A new fixed point theorem and its applications,* Proc. Amer. Math. Soc., **125** (1997), 1779-1783.

[21] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibrium in linear topological spaces,* J. Math. Econom. **12** (1983), 233-245.

[22] X. Z. Yuan, *The Study of Minimax Inequalities and Applications to Economies and Variational Inequalities,* Memoirs of the American Society, **132**, (1988).

[23] L. A. Zadeh, *Fuzzy sets,* Inform and Control **8** (1965), 338-353.

[24] X. Zheng, *Approximate selection theorems and their applications,* J. Math. An. Appl. **212,** 88-97 (1997)